中国科学院大学
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CS101

Logic Thinking

Predicative Logic

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Outline

- Foundation of logic
  - Propositional Logic
  - Predicative Logic
- Automata and Turing Machines
- Power and Limitation of Computing
  - Mechanical Theorem Proving
  - Church-Turing Hypothesis

1.2 PREDICATIVE LOGIC

Quantifiers

- Universal quantifier ∀
  - Everyday, there are 24 hours.
- Existential quantifier ∃
  - There exists some prime number which is even.

- "Not-all" and "all-not"
  - $\forall x\, (\neg P(x)),\quad \exists x(\neg P(x))$
  - $\exists x\big(\neg P(x)\big) = \neg(\forall x\, \big(P(x)\big))$

Domain and Order

- Domain
  - For any natural number, either it is an even number, or its successor is an even number.
  - $\forall n\ [\mathrm{Even}(n) \vee \mathrm{Even}(n+1)]$, where the predicate Even(n) means n is an even number
  - $\forall n \in \mathbb{N}\ [\mathrm{Even}(n) \vee \mathrm{Even}(n+1)]$
  - Difference between the above expressions
- Order
  - $\forall\, x, \exists\, y\ (y = x + 1)$
  - $\exists\, y, \forall\, x\ (y = x + 1)$
  - Difference between the above expressions

What is Logic Thinking?

- "There exist infinitely many prime numbers"
- How to describe it rigorously?
  - Key point: infinitely

$$\forall n, \exists m, [(m > n) \wedge (\text{Prime } (m))]$$

$$\forall n, \exists m, \forall p, q \; [(m > n) \wedge (p, q > 1 \rightarrow pq \neq m)]$$

- How to prove?

Euclid of Alexandria
Elements

- There exist infinitely many prime numbers

- Method 1
  - For any given n, we try to find a prime number larger than n, …
  - $\forall n, \exists m, [(m > n) \wedge (\text{Prime }(m))]$
- Method 2 (proof from Euclid)
  - If there are finite prime numbers, …., we reach contradiction.
  - $\neg(\exists n, \forall m, \big[(m > n) \rightarrow \neg\big(\text{Prime }(m)\big)\big])$

Example 2

- For any $n > 2$, there is no non-trivial solution for equation $x^n + y^n = z^n$.
  - Key point: non-trivial solution
  - $\forall a, b, c, n\ [(abc \neq 0) \land (n > 2) \rightarrow a^n + b^n \neq c^n]$

Thinking Problem

- There are infinitely many pairs of twin primes.

- ($3n + 1$ conjecture) For any positive integer n, if it is an odd number, multiply by 3 and add 1, if it is an even number, divide by 2. Repeat this process, and finally get 1.

3n + 1 conjecture

- $6 \rightarrow 3 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$
- $15 \rightarrow 46 \rightarrow 23 \rightarrow 70 \rightarrow 35 \rightarrow 106 \rightarrow 53 \rightarrow 170 \rightarrow 85 \rightarrow 256 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

More examples

- Four-color theorem: Any planar graph can be four-colored, making any two adjacent vertices different colors.
  - $\forall planar\ graph\ G = (V, E), \exists c: V \to \{1,2,3,4\}\ \forall (u, v) \in E, [c(u) \neq c(v)]$

- Satisfaction problem: Given the CNF formula φ, is there an assignment that makes this CNF true?
  - $\exists A: \{x_1, x_2, \ldots, x_n\} \to \{0,1\}, [\phi(x_1, \ldots, x_n) = 1]$